15.2. Double integrals over general regions : properties

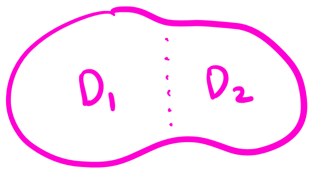
Prop Let $f(x,y)$ and $g(x,y)$ be functions on a domain D.

(1) $\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$.

(2) $\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$ for any number c.

(3) If D is split into subdomains D_1 and D_2, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

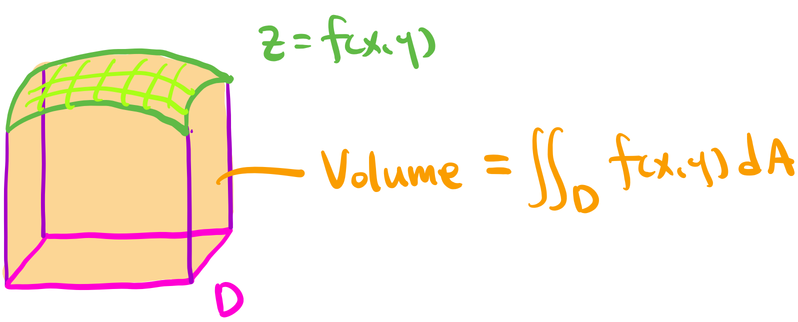


(4) $\text{Area}(D) = \iint_D 1\,dA$

(5) The average value of $f(x,y)$ on D is given by

$$\frac{1}{\text{Area}(D)}\iint_D f(x,y)\,dA$$

(6) $\iint_D f(x,y)\,dA$ equals the (signed) volume of the solid under the graph $z=f(x,y)$ and above D.



Recall: If $f(x)$ is odd (i.e. $f(-x) = -f(x)$ for all x), then

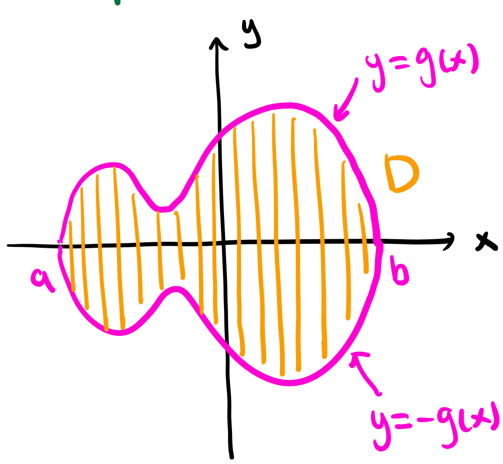
$$\int_{-a}^{a} f(x)\,dx = 0 \text{ for any number } a.$$

<u>Prop</u> (Double integrals and symmetry).

Let $f(x,y)$ be a function on a domain D.

(1) If D is symmetric about the x-axis while $f(x,y)$ is odd with respect to y, then $\iint_D f(x,y)\,dA = 0$.

* Explanation:



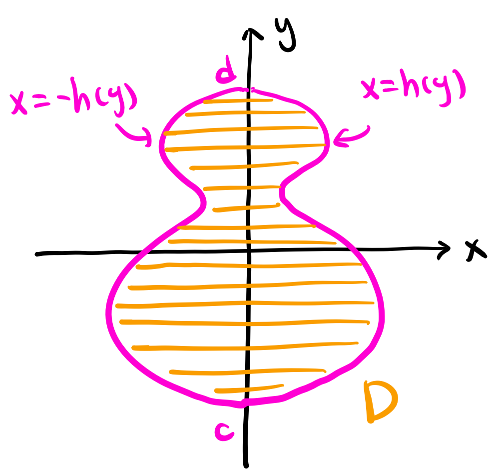
D: $a \le x \le b$, $-g(x) \le y \le g(x)$

$$\iint_D f(x,y)\,dA = \int_a^b \underbrace{\int_{-g(x)}^{g(x)} f(x,y)\,dy}_{=0}\,dx = 0$$

where the inner integral is zero for $f(x,y)$ being odd with respect to y.

(2) If D is symmetric about the y-axis while $f(x,y)$ is odd with respect to x, then $\iint_D f(x,y)\,dA = 0$.

* Explanation:



D: $c \le y \le d$, $-h(y) \le x \le h(y)$

$$\iint_D f(x,y)\,dA = \int_c^d \underbrace{\int_{-h(y)}^{h(y)} f(x,y)\,dx}_{=0}\,dy = 0$$

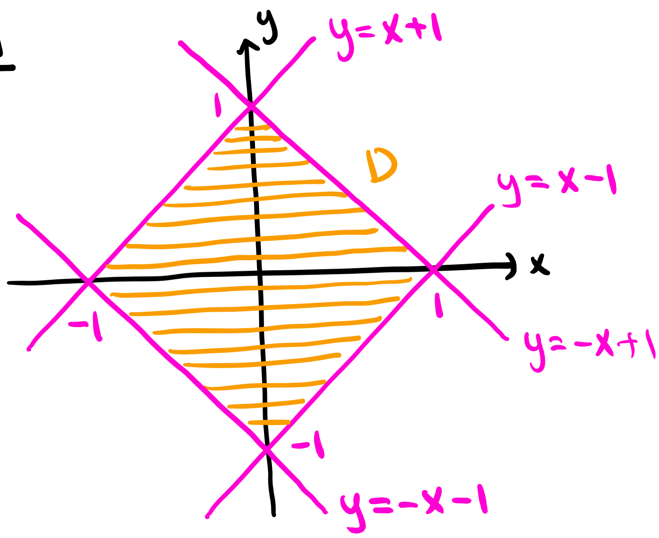
where the inner integral is zero for $f(x,y)$ being odd with respect to x.

Ex Let D be the region bounded by the curves

$y = x+1$, $y = x-1$, $y = -x+1$, and $y = -x-1$.

(1) Evaluate $\iint_D \sin(x)\cos(y)\,dA$.

Sol



D is symmetric about both the x-axis and the y-axis.

$$\Rightarrow \iint_D \underbrace{\sin(x)\cos(y)}_{\text{odd w.r.t } x}\,dA = \boxed{0}$$

(2) Find the volume of the solid under the surface

$z = x^3y^2 + 2x^2y^5 + 4$ over D.

Sol Volume $= \iint_D x^3y^2 + 2x^2y^5 + 4\,dA$

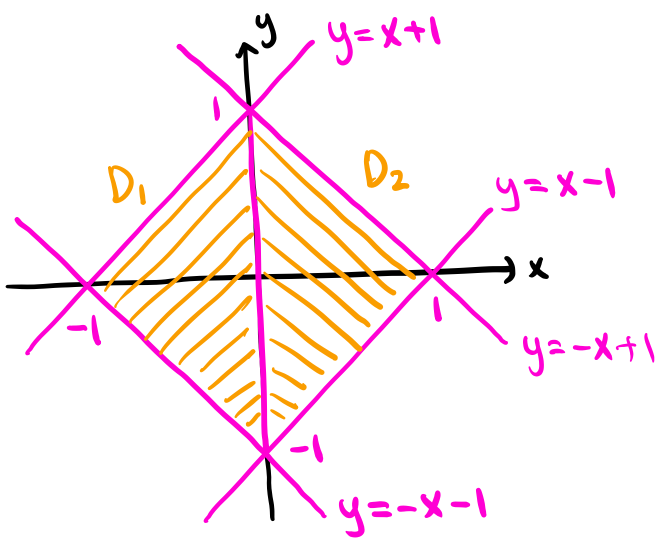
$$= \iint_D \underbrace{x^3y^2}_{\text{odd w.r.t. } x}\,dA + 2\iint_D \underbrace{x^2y^5}_{\text{odd w.r.t. } y}\,dA + 4\iint_D 1\,dA$$

$$= 0 + 0 + 4\underbrace{\text{Area}(D)}_{=2} = \boxed{8}$$

(3) Evaluate $\iint_D x^2 \, dA$

Sol The function x^2 is not odd with respect to x or y, but is even with respect to x (and y).

We divide D along the y-axis as follows



$$\Rightarrow \iint_{D_1} x^2 \, dA = \iint_{D_2} x^2 \, dA$$

(x^2 is even with respect to y)

$$\Rightarrow \iint_D x^2 \, dA = \iint_{D_1} x^2 \, dA + \iint_{D_2} x^2 \, dA = 2\iint_{D_1} x^2 \, dA.$$

$$D_1 = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 0, \ -x-1 \le y \le x+1\}$$

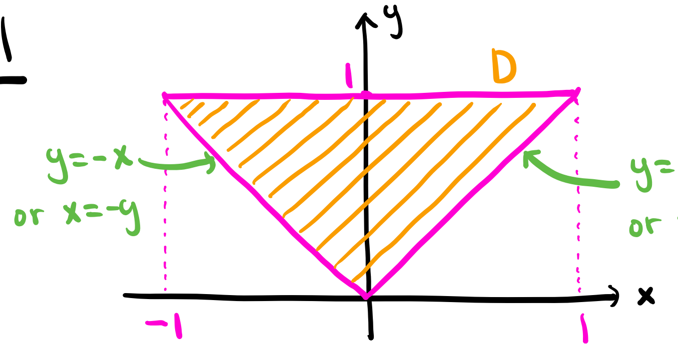
$$\iint_{D_1} x^2 \, dA = \int_{-1}^{0} \int_{-x-1}^{x+1} x^2 \, dy \, dx = \int_{-1}^{0} x^2(2x+2) \, dx$$

$$= \int_{-1}^{0} 2x^3 + 2x^2 \, dx = \left(\frac{1}{2}x^4 + \frac{2}{3}x^3\right)\Bigg|_{x=-1}^{x=0} = \frac{1}{6}$$

$$\Rightarrow \iint_D x^2 \, dA = 2 \cdot \frac{1}{6} = \boxed{\frac{1}{3}}$$

Ex Find the volume of the solid under the surface $z = 2y^2 e^{xy}$ and over the triangular region D with vertices at $(-1,1)$, $(0,0)$, $(1,1)$.

Sol



The volume is equal to $\iint_D 2y^2 e^{xy}\, dA$.

$D = \{ (x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ -y \le x \le y \}$.

$$\iint_D 2y^2 e^{xy}\, dA = \int_0^1 \int_{-y}^{y} 2y^2 e^{xy}\, dx\, dy$$

$$= \int_0^1 2y\, e^{xy} \Big|_{x=-y}^{x=y} dy$$

$$= \int_0^1 2y e^{y^2} - 2y e^{-y^2}\, dy$$

$$= \left(e^{y^2} + e^{-y^2}\right)\Big|_{y=0}^{y=1} = \boxed{e + e^{-1} - 2}$$

Note Even though D is symmetric about the y-axis, you cannot use symmetry here because the function $2y^2 e^{xy}$ has no symmetry.